CHARACTERISTIC FUNCTIONAL OF A
PROBABILITY MEASURE ABSOLUTELY CONTINUOUS
WITH RESPECT TO A GAUSSIAN RADON MEASURE*

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ADA 149482

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED	1b. RESTRICTIVE MARKINGS

2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION/AVAILABILITY OF REPORT
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE	Approved for Public Release: Distribution Unlimited

4. PERFORMING ORGANIZATION REPORT NUMBER(S)	5. MONITORING ORGANIZATION REPORT NUMBER(S)

6a. NAME OF PERFORMING ORGANIZATION Department of Statistics	6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION

6c. ADDRESS (City, State and ZIP Code) University of North Carolina Chapel Hill, North Carolina 27514	7b. ADDRESS (City, State and ZIP Code)

8a. NAME OF FUNDING/SPONSORING ORGANIZATION Office of Naval Research	8b. OFFICE SYMBOL (If applicable)	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER N00014-81-K-0373

8c. ADDRESS (City, State and ZIP Code) Statistics & Probability Program Arlington, VA 22217	10. SOURCE OF FUNDING NOS.			

PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT NO.
NR	042	269	SRO 105

11. TITLE (Include Security Classification)
Characteristic Functional of a Probability . . .

12. PERSONAL AUTHOR(S)
Hiroshi Sato

13a. TYPE OF REPORT TECHNICAL	13b. TIME COVERED FROM _____ TO _____	14. DATE OF REPORT (Yr., Mo., Day) Aug. 1984	15. PAGE COUNT 31

16. SUPPLEMENTARY NOTATION

17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB. GR.	Gaussian Radon measures, absolute continuity, characteristic functionals, convolution

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

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The aim of this paper is to give an answer to this problem in the case where μ is a Gaussian Radon measure. We shall define a Fourier transform, establish the inversion formula, and then give a necessary and sufficient condition for μ_1 to be absolutely continuous with respect to μ based on the characteristic functional. As applications, for the convolution $\mu_1 = \mu * \nu$, where ν is a Randon measure on E, we shall give some concrete sufficient conditions on ν for $\mu * \nu \ll \mu$.

TITLE CONT.: Measure Absolutely Continuous with Respect to a Gaussian Radon Measure.

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☒ DTIC USERS ☐	21. ABSTRACT SECURITY CLASSIFICATION

22a. NAME OF RESPONSIBLE INDIVIDUAL C.R. Baker	22b. TELEPHONE NUMBER (Include Area Code) (919) 962-2189	22c. OFFICE SYMBOL

DD FORM 1473, 83 APR   EDITION OF 1 JAN 73 IS OBSOLETE.

ABSTRACT

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The aim of this paper is to give an answer to this problem in the case where μ is a Gaussian Radon measure. We shall define a Fourier transform, establish the inversion formula, and then give a necessary and sufficient condition for μ_1 to be absolutely continuous with respect to μ based on the characteristic functional. As applications, for the convolution $\mu_1 = \mu * \nu$, where ν is a Radon measure on E, we shall give some concrete sufficient conditions on ν for $\mu * \nu \ll \mu$.

§1.   INTRODUCTION AND NOTATIONS.

Let E be a locally convex Hausdorff real topological linear space, E*
be the topological dual space of E, $<x,\xi>$ be the canonical bilinear form on
E×E*, $B(E)$ be the Borel field of E, and $C(E,E*)$ be the σ-algebra generated by
$\{<x,\xi>;\ \xi\epsilon E*\}$.   Every Radon measure μ on $(E,B(E))$ is uniquely determined by the
characteristic functional

$$\tilde{\mu}(\xi) = \int_E e^{i<x,\xi>}d\mu(x),\quad \xi\epsilon E*.$$

Therefore a Radon measure μ_1 is absolutely continuous with respect to μ
(denoted by $\mu_1 << \mu$) on $(E,B(E))$ if and only if $\mu_1 << \mu$ on $(E,C(E,E*)$.   If μ_1
and μ are mutually absolutely continuous, we denote it by $\mu_1 \sim \mu$.

Let μ be a Gaussian Radon measure on $(E,B(E))$.   Then without loss of
generality we may assume that the mean is zero (C. Borell [3]) and E is the
minimal closed linear subspace of probability 1.   Let H be the reproducing
kernel Hilbert space of μ with norm $|\xi|$ and inner product (ξ,η).   Then H is
separable (H. Sato and Y. Okazaki [6], C. Borell [3]), and, identifying H*
with H, we have E* $\subset$ H $\subset$ E, where E* is dense in H. The characteristic functional
of μ is $\tilde{\mu}(\xi) = \exp[-\frac{1}{2}|\xi|^2]$.

A subset A of E is called <u>full</u> iff $A\epsilon B(E)$ and μ(A)=1.   A <u>random linear map</u>
T of E to a topological linear space $(F,B(F))$ is a measurable linear map defined
on a full linear subspace of E.   Every random linear map T has an explicit value
on H since H is included in every full linear subspace of E (H. Sato and Y.
Okazaki [6]).   A real valued random linear map is called a <u>random linear</u>
<u>functional</u>.   In §2 we shall show that every $\xi\epsilon H$ is extended to a random linear
functional denoted by (x,ξ) and that every finite dimensional orthogonal
projection P of H is extended to a random linear map of E onto PH, which is
again denoted by P.

The collection of all finite dimensional orthogonal projections of H is denoted by P. A sequence $\{P_n\}$ in P is called an <u>approximation sequence</u> iff $x = \lim_n P_n x$ for every $x \in H$ and $P_n H \subset P_{n+1} H$, $n \in \mathbf{N}$.

The characteristic functional of a Radon measure μ_1 on E is called μ-<u>continuous</u> iff $\mu_1 \ll \mu$, and μ-<u>equivalent</u> iff $\mu_1 \sim \mu$. Let $\tilde{\mu}_1(\xi)$ be a μ-continuous characteristic functional. Then there exists the Radon-Nikodym derivative $p(x) = \dfrac{d\mu_1}{d\mu}(x)$, a.e. (μ), and we have

$$\tilde{\mu}_1(\xi) = \mathbf{E}_x[e^{i(x,\xi)}p(x)], \quad \xi \in H,$$

where $\mathbf{E}_x[\ ]$ is the mathematical expectation with respect to μ in the variable x. Therefore our problem is reduced to characterize the image of

$$L_+^1(\mu) = \{\phi \in L^1(\mu); \ \phi(x) \geq 0, \ \text{a.s.}(\mu)\}$$

under the above transformation. However it is more convenient for our purpose to define the Fourier transform F on $L^1(\mu)$ by

$$\hat{\phi}(\xi) = (F\phi)(\xi) = e^{\frac{1}{2}|\xi|^2} \mathbf{E}_x[e^{i(x,\xi)}\phi(x)], \quad \xi \in H, \quad \phi \in L^1(\mu).$$

Then our problem is equivalent to characterize the image of $L_+^1(\mu)$ under F.

In §2 of this paper we prove the Riemann-Lebesgue property for a μ-continuous characteristic functional. In §3 we discuss the tame functionals. In §4, combining the idea of the "measurable function" of L. Gross [4] and the theory of the classical Fourier transform, we establish the inversion formula for F (Theorem 4), under the assumption of the integrability, give a necessary and sufficient condition for a continuous functional on H to be the Fourier image of a μ-integrable functional (Theorem 5), give a necessary and sufficient condition for a functional on H to be a μ-continuous characteristic function (Theorem 6), and then give an application to the Wiener measure (Theorem 7). In §5 we discuss the convolution $\mu_1 = \mu*\nu$, where ν

is a Radon measure on E.   Since μ is H-quasi-invariant, $\mu*\nu \ll \mu$ if $\nu(H)=1$.
We give some concrete sufficient conditions on ν for $\mu*\nu \ll \mu$, including the
case $\nu(H)=0$.

Throughout the paper every non-negative definite functional f is assumed
to be normalized, i.e. f(0)=1, except for Lemma 3,4, and every Radon measure
is a probability measure.   The measure $d\lambda(t)$ on $\mathbb{R}^d$ denotes the modified
Lebesgue measure

$$d\lambda(t) = d\lambda(t_1,t_2,\ldots,t_d) = (2\pi)^{-d/2}dt_1 dt_2 \ldots dt_d.$$

We also use the following notations.

$$C(\xi) = \exp[-\frac{1}{2}|\xi|^2], \quad \overline{C}(\xi) = \exp[\frac{1}{2}|\xi|^2],$$

$\mathbb{R}$ = real numbers, $\mathbb{C}$ = complex numbers,

$\mathbb{R}^d$ = d-dimensional Euclidean space,

$\mathbb{N}$ = natural numbers, $\mathbb{N}_0$ = non-negative integers.

§2.   RIEMANN-LEBESGUE PROPERTY

Let μ_1 be a Radon measure on E and $\tau(\mu_1)$ be the topology on E*
induced by the convergence in probability μ_1. Then, if $\mu_1 \ll \mu$, $\tau(\mu_1)$ is weaker
than $\tau(\mu)$. On the other hand $\tau(\mu)$ is equivalent to the topology of H.
Therefore if $\mu_1 \ll \mu, \tilde{\mu}_1(\xi)$ is continuous in the topology of H so that extended
to H. In this paper we only consider the non-negative definite continuous
functionals on H.

For every ξ in H, since E* is dense in H, there exists a sequence $\{\xi_n\}$
in E* such that $\lim_n |\xi - \xi_n| = 0$. Then, since $\{<x,\xi_n>\}n \in \mathbb{N}$ is a Caucy sequence
in $\tau(\mu)$, $<x,\xi_n>$ converges to a real random variable $\xi(x)$ in $\tau(\mu)$. Let
$\{<x,\xi_{n_k}>\}k \in \mathbb{N}$ be a subsequence which converges almost surely to $\xi(x)$ and
define $(x,\xi) = \lim_k <x,\xi_{n_k}>$. Then (x,ξ) is a random linear functional on (E,μ).
Therefore the definition of (x,ξ) depends on the choice of the sequence $\{\xi_{n_k}\}$
but it is easy to show that (x,ξ) is defined uniquely up to μ-null sets so
that the distribution of (x,ξ) is determined uniquely. (x,ξ) obeys to a
Gaussian distribution of mean zero and variance $|\xi|^2$ (denoted by $N(0,|\xi|^2)$),
and (x,ξ) and (x,η), $\xi,\eta \in H$, are independent if and only if $(\xi,\eta)=0$.

For every P in $\mathcal{P}$, let $\{e_k: 1 \le k \le d\}$ be a CONS (complete orthonormal system)
of PH. Then obviously

$$\overset{o}{P}x = \sum_{k=1}^{d} (x,e_k)e_k$$

defines a random linear map of E onto PH. The definition of $\overset{o}{P}$ depends on
the choice of the CONS but is unique up to μ-null sets so that the distri-
bution of $\overset{o}{P}x$ is determined uniquely. We denote $\overset{o}{P}$ again by P and remark
that we have

$$(x,P\xi) = (Px,P\xi) = (Px,\xi)$$

almost surely on $(E \times E, \mu \times \mu)$.

Let $\mathcal{B}(P)$ be the σ-algebra on E generated by $\{(x,\xi); \xi \in PH\}$, and for every approximation sequence $\{P_n\}$ in P, let $\underset{n}{\vee}\mathcal{B}(P_n)$ be the σ-algebra generated by $\underset{n}{\cup}\mathcal{B}(P_n)$. Then for every $C(E,E^*)$-measurable function φ on E there exists a $\underset{n}{\vee}\mathcal{B}(P_n)$-measurable function ϕ_1 such that $\phi(x) = \phi_1(x)$, a.s. (μ). Therefore it is sufficient to show the absolute continuity on $\underset{n}{\vee}\mathcal{B}(P_n)$ for an approximation sequence $\{P_n\}$ in P.

For every φ in $L^1(\mu)$ and every P in P we denote the conditional expectation with respect to $\mathcal{B}(P)$ by

$$(P\phi)(x) = \mathbb{E}_x[\phi | \mathcal{B}(P)].$$

Let $\{e_k : 1 \le k \le d\}$ be a CONS of PH (so that d is the dimension of PH). Then there exists a Borel function $(P\phi)^*(t) = (P\phi)^*(t_1, t_2, \ldots, t_d)$ on $\mathbb{R}^d$ such that

$$(P\phi)(x) = (P\phi)^*((x,e_1), (x,e_2), \ldots, (x,e_d)), \quad \text{a.s.} (\mu).$$

THEOREM 1. Let φ be in $L^1(\mu)$ and define

$$\tilde{\phi}(\xi) = \mathbb{E}_x[e^{i(x,\xi)}\phi(x)], \quad \xi \in H.$$

Then we have the following.

(1) $\tilde{\phi}$ is bounded and uniformly continuous on H.

(2) For every approximation sequence $\{P_n\}$ in P

$$\lim_n \sup_{\xi \in H} |\tilde{\phi}(\xi) - \widetilde{(P_n\phi)}(\xi)| = 0$$

(3) (Riemann-Lebesque property)

$$\lim_{|\xi| \to \infty} \tilde{\phi}(\xi) = 0.$$

Proof (1) is trivial.

(2) Let $\{P_n\}$ be an approximation sequence in P. Then $P_n\phi = \mathbb{E}_x[\phi | \mathcal{B}(P_n)]$ converges to φ in $L^1(\mu)$ (J. Neveu [5] Proposition IV-2-3) so that we have

$$\sup_{\xi \in H} |\tilde{\phi}(\xi) - \widetilde{(P_n\phi)}(\xi)|$$

$$= \sup_{\xi \in H} |\mathbb{E}_x[e^{i(x,\xi)}(\phi(x) - (P_n\phi)(x)]|$$

$$\leq \mathbb{E}_x[|\phi(x) - (P_n\phi)(x)|] \to 0 \quad \text{as } n \to +\infty,$$

(3)   For every $\varepsilon > 0$ by (2) there exists P in $\mathcal{P}$ such that $\sup_{\xi \in H} |\tilde{\phi}(\xi) - \widetilde{(P\phi)}(\xi)| < \frac{1}{2}\varepsilon$.

On the other hand we have

$$|\widetilde{(P\phi)}(\xi)| = |\mathbb{E}_x[e^{i(x,\xi)}(P\phi)(x)]|$$

$$= |\mathbb{E}_x[e^{i(x,\xi)}(P\phi)^*((x,e_1),(x,e_2),\ldots,(x,e_d))]|$$

$$= C((I-P)\xi) \, J(\xi),$$

$$J = J(\xi) = \int_{R^d} e^{i\Sigma_k t_k(\xi,e_k)} (P\phi)^*(t_1,t_2,\ldots,t_d) \, C(t_1,t_2,\ldots,t_d) d\lambda(t_1,\ldots,t_d),$$

where I is the identity operator of H, $\{e_k : 1 \leq k \leq d\}$ is a CONS of PH. Then we can easily prove that $\lim_{|\xi| \to \infty} C((I-P)\xi)J(\xi) = 0$ by the classical Riemann-Lebesgue theorem and the boundedness of $C(\xi)$ and $J(\xi)$.

From now on we denote the integral $J(\xi)$ symbolically by

$$J(\xi) = \int e^{i(Px,P\xi)}(P\phi)(Px)C(Px)d\lambda(Px).$$

§3.   TAME FUNCTIONALS

A μ-measurable functional $\phi = \phi(x)$ on E is called a <u>tame functional</u> if there exists P in P such that $\phi(x) = \phi(Px)$, a.s. (μ), and a functional $\psi = \psi(\xi)$ on H is called a <u>tame functional</u> if there exists P in P such that $\psi(\xi) = \psi(P\xi)$ for every $\xi \in H$.   It is obvious that every continuous tame functional on H is extended to a μ-measurable tame functional on E.

The following lemma is proved in S. Bochner and K. Chandrasekharan [2], Theorem 37, 38.

LEMMA 1.   <u>Define the classical Fourier transform by</u>

$$\tilde{f}(\alpha) = \int_{R^d} e^{i(\alpha, t)} f(t) d\lambda(t), \quad \alpha \in \mathbb{R}^d,$$

<u>for</u> $f \in L^1(\mathbb{R}^d, d\lambda)$.   <u>Then we have</u>

(1) $f(t) = \lim_{\epsilon \downarrow 0} \int_{R^d} e^{-i(t, \alpha)} \tilde{f}(\alpha) C(\sqrt{\epsilon}\alpha) d\lambda(\alpha)$, a.s.$(d\lambda)$.

(2)   <u>In particular if</u> $\tilde{f}(\alpha)$ <u>is</u> $d\lambda(\alpha)$-<u>integrable, then</u>

$$f(t) = \int_{R^d} e^{-i(t, \alpha)} \tilde{f}(\alpha) d\lambda(\alpha), \quad a.s.(d\lambda).$$

On the other hand we have the following lemma.

LEMMA 2.   <u>Let</u> ϕ <u>be a</u> μ-<u>integrable tame functional on</u> E <u>such that</u> $\phi(x) = \phi(Px)$, a.s.(μ), <u>for a</u> P <u>in</u> P.   <u>Then</u> $\hat{\phi} = F\phi$ <u>has the representation</u> $\hat{\phi}(\xi) = \hat{\phi}(P\xi)$ <u>for all</u> ξ <u>in</u> H, <u>and extends to a tame functional on</u> E.

Proof.   Assume that ϕ satisfies the hypothesis.   Then we have

$$\hat{\phi}(P\xi) = \overline{C}(P\xi) \, E_x [e^{i(x, P\xi)} \phi(x)]$$

$$= \overline{C}(P\xi) \, E_x [e^{i(x, P\xi)} \phi(Px)]$$

$$= \overline{C}(\xi) \, E_x (e^{i(x, \xi)} \phi(Px)]$$

$$= \overline{C}(\xi) \, E_x [e^{i(x, \xi)} \phi(x)] = \hat{\phi}(\xi), \quad \xi \in H.$$

Q.E.D.

For a μ-integrable function ψ we define

$$(\overline{F}\psi)(\xi) = \overline{C}(\xi)\,\mathbb{E}_x[e^{-i(x,\xi)}\psi(x)], \quad \xi\epsilon H.$$

If ψ is a tame functional, $\overline{F}\psi$ is also a tame functional on H and can be extended to a tame functional on E.

THEOREM 2. Let ψ be a continuous tame functional on H which is μ-integrable on E. Then ψ is the Fourier transform of a μ-integrable tame functional on E if and only if $\phi = \overline{F}\psi$ is μ-integrable.

Moreover if such ϕ is μ-integrable, we have $\psi = F\phi$.

Proof. Assume that $\psi(\xi) = \psi(P\xi)$, $\xi\epsilon H$, for a P in P and $\phi = \overline{F}\psi$ is μ-integrable. Then we have

$$\begin{aligned}
\phi(x) &= \phi(Px) \\
&= \overline{C}(Px)\,\mathbb{E}_\xi[e^{-i(\xi,Px)}\psi(\xi)] \\
&= \overline{C}(Px)\,\mathbb{E}_\xi[e^{-i(P\xi,Px)}\psi(P\xi)]
\end{aligned}$$

so that

$$\begin{aligned}
C(Px)\phi(Px) &= \mathbb{E}_\xi[e^{-i(Px,P\xi)}\psi(P\xi)] \\
&= \int e^{-i(Px,P\xi)}\psi(P\xi)C(P\xi)d\lambda(P\xi).
\end{aligned}$$

This shows that $C(Px)\phi(Px)$ is the classical Fourier inverse transform of $C(P\xi)\psi(P\xi)$. If $\phi(x) = \phi(Px)$ is μ-integrable, then $C(Px)\phi(Px)$ is $d\lambda(Px)$-integrable and we have by Lemma 1(2)

$$\psi(P\xi)C(P\xi) = \int e^{i(Px,P\xi)}C(Px)\phi(Px)d\lambda(Px), \quad \text{a.s.}(d\lambda),$$

so that

$$\begin{aligned}
\psi(\xi) = \psi(P\xi) &= \overline{C}(P\xi)\,\mathbb{E}_x[e^{i(Px,P\xi)}\phi(Px)] \\
&= \overline{C}(\xi)\,\mathbb{E}_x[e^{i(x,\xi)}\phi(Px)] \\
&= \overline{C}(\xi)\,\mathbb{E}_x[e^{i(x,\xi)}\phi(x)] = (F\phi)(\xi), \quad \text{a.s.}(\mu).
\end{aligned}$$

The necessity is a simple corollary of Lemma 1.

$$\text{Q.E.D.}$$

We estimate the Fourier transform of some typical tame functionals.

EXAMPLE 1.

$$\phi(x) \qquad\qquad \hat{\phi}(\xi)$$

(1) $e^{\alpha(x,\eta)}$, $\quad e^{\frac{1}{2}\alpha^2|\eta|^2+\alpha i(\xi,\eta)}$, $\quad \alpha\in\mathbb{C}$, $\eta\in H$,

(2) $e^{i(x,\eta)}$, $\quad e^{-(\xi,\eta)-\frac{1}{2}|\eta|^2}$, $\quad \eta\in H$

(3) $(x,\eta)^n$, $\quad i^n h_n((\xi,\eta))$, $\quad \eta\in H$, $|\eta|=1$, $n\in\mathbb{N}_0$,

(4) $h_n((x,\eta))$, $i^n(\xi,\eta)^n$, $\quad \eta\in H$, $|\eta|=1$, $n\in\mathbb{N}_0$,

where $h_n(s) = (-1)^n e^{\frac{s^2}{2}} \dfrac{d^n}{ds^n} e^{-\frac{s^2}{2}}$, $\quad n\in\mathbb{N}_0$.

The proof of the following lemma was given by a personal communication from J. Bourgain.

LEMMA 3.

Let ν be a finite measure on $\mathbb{R}^d$ which is singular to $d\lambda$. Then we have

$$\lim_{\varepsilon\to 0} (\varepsilon)^{-\frac{d}{2}} \int_{\mathbb{R}^d} C\left(\frac{s-t}{\sqrt{\varepsilon}}\right) d\nu(s) = 0, \quad a.s. \ (d\lambda(t)).$$

Proof. Without loss of generality we may assume that ν is a probability measure. Since ν is singular to $d\lambda$, there exists a Borel subset N of $\mathbb{R}^d$ such that $\lambda(N) = 0$ and $\nu(N) = 1$.

For every $\varepsilon>0$ define

$$f_\varepsilon(t) = (\varepsilon)^{-\frac{d}{2}} \int_{\mathbb{R}^d} C\left(\frac{s-t}{\sqrt{\varepsilon}}\right) d\nu(s)$$

$$= (\varepsilon)^{-\frac{d}{2}} \int_N C\left(\frac{s-t}{\sqrt{\varepsilon}}\right) d\nu(s), \quad t\in\mathbb{R}^d,$$

and for every $\delta>0$

$$A = A(\delta) = \{t\in\mathbb{R}^d; \ \limsup_{\varepsilon\to 0} f_\varepsilon(t) \geq \delta\}.$$

Then for every t in A there exists a sequence of positive numbers $\varepsilon_k = \varepsilon_k(t) \to 0$ and $f_{\varepsilon_k}(t) \geq \delta$ for all $k \in \mathbb{N}$.

Fix any t in A and any $\varepsilon > 0$, and define

$$B_k = \{s \in \mathbb{R}^d; \ 2k\varepsilon \leq |s-t|^2 < 2(k+1)\varepsilon\}, \quad k \in \mathbb{N}_0.$$

Then we have

$$\sqrt{\varepsilon}^d f_\varepsilon(t) = \sum_k \int_{B_k} C(\frac{s-t}{\sqrt{\varepsilon}}) d\nu(s) \geq \sqrt{\varepsilon}^d \delta$$

so that

$$\delta\sqrt{\varepsilon}^d \leq \sum_k e^{-k} \nu(B_k).$$

Let $\ell = \ell(\varepsilon)$ be the minimal number in $\mathbb{N}_0$ greater than $1 - \log \delta - \frac{d}{2} \log \varepsilon - \log \frac{e-1}{2e}$. Then we have

$$\sum_{k \geq \ell} e^{-k} = \frac{e^{-\ell}}{1-e^{-1}} \leq \frac{\delta}{2}\sqrt{\varepsilon}^d,$$

so that

$$\delta\sqrt{\varepsilon}^d \leq \sum_k e^{-k} \nu(B_k)$$

$$\leq \sum_{k < \ell} e^{-k} \nu(B_k) + \sum_{k \geq \ell} e^{-k}$$

$$\leq \sum_{k < \ell} e^{-k} \nu(B_k) + \frac{\delta}{2}\sqrt{\varepsilon}^d,$$

and

$$\sum_{k < \ell} e^{-k} \nu(B_k) \geq \frac{\delta}{2}\sqrt{\varepsilon}^d.$$

Define $L = \sqrt{e}/(\sqrt{e}-1) = \sum_{k=0}^{\infty} e^{-\frac{k}{2}}$ and $M = \delta\sqrt{\varepsilon}^d/2L$. Then there exists $k(\varepsilon) \leq \ell = \ell(\varepsilon)$ such that $\nu(B_{k(\varepsilon)}) \geq Me^{k(\varepsilon)}$. In fact assume the contrary. Then for every $k \leq \ell$ we have $\nu(B_k) < Me^{\frac{k}{2}}$ so that

$$\sum_{k<\ell} e^{-k} \nu(B_k) < \sum_{k<\ell} e^{-\frac{k}{2}} M$$

$$< L \frac{\delta\sqrt{\varepsilon}^d}{2L} = \frac{\delta}{2} \sqrt{\varepsilon}^d,$$

which is a contradiction.

Define for every $k \geq 0$

$$D_k = \{s \in \mathbf{R}^d; \ |s-t|^2 \leq 2(k+1)\varepsilon\}.$$

Then we have $\lambda(D_k) = \gamma\sqrt{(k+1)\varepsilon}^d$, where γ is a positive constant independent of t, ε and k, so that

$$\nu(D_{k(\varepsilon)}) \geq \nu(B_{k(\varepsilon)}) \geq e^{\frac{k(\varepsilon)}{2}} \frac{\delta}{2L} \sqrt{\varepsilon}^d.$$

Since

$$\beta = \inf_{k>0} \frac{e^{\frac{k}{2}}\delta}{2L} \Big/ \theta(k+1)^{\frac{d}{2}} > 0,$$

we have

$$\nu(D_{k(\varepsilon)}) \geq \beta\gamma\sqrt{(k+1)\varepsilon}^d = \beta\lambda(D_{k(\varepsilon)}).$$

On the other hand, we have $k(\varepsilon) \leq \ell(\varepsilon) = 2 - \log \delta - \frac{d}{2} \log \varepsilon - \log \frac{e-1}{2e}$ so that

$$\lambda(D_{k(\varepsilon)}) = \gamma\sqrt{(k(\varepsilon)+1)\varepsilon}^d$$

$$\leq \gamma\sqrt{(\ell(\varepsilon)+1)\varepsilon}^d \to 0 \text{ as } \varepsilon \to 0.$$

Thus we have proved that for every t in A and every $r>0$ there exists a closed ball D with center t and radius less than r such that $\nu(D) \geq \beta\lambda(D)$.

Assume that $b = \lambda(A) > 0$. Then there exists a compact subset K of N such that $\nu(K) \geq 1 - \frac{\beta}{2} b$ and we have $\lambda(K) \leq \lambda(N) = 0$. Define $A' = A \backslash K$. Then we can choose a sequence $\{D_n\}$ of closed balls such that

$$\bigcup_n D_n \supset A', \ D_k \cap K = \emptyset, \ D_k \cap D_\ell = \emptyset (K \neq \ell), \quad k,\ell \in \mathbf{N}.$$

Then we have

$$\nu(A') = \nu(N \cap \bigcup_n D_n) = \nu(\bigcup_n D_n)$$

$$\geq \beta\lambda(\bigcup_n D_n) = \beta\{\lambda(\bigcup_n D_n) + \lambda(K)\}$$

$$\geq \beta\lambda(A) \geq \beta b.$$

On the other hand, by the assumption we have $\nu(K) \geq 1 - \frac{1}{2}\beta b$ so that

$$\nu(A) = \nu(A') + \nu(K)$$

$$\geq \beta b + 1 - \frac{1}{2}\beta b$$

$$= 1 + \frac{1}{2}\beta b > 1.$$

This is a contradiction.

Thus we have proved that $\lambda(A(\delta)) = 0$ for every $\delta > 0$, which implies

$$\lambda(t \in \mathbb{R}^d : \limsup_\epsilon f_\epsilon(t) > 0) = 0.$$

$$Q.E.D.$$

As an application of Lemma 3 we have the following lemma.

LEMMA 4.

Let $f = f(\alpha)$ be a complex valued function on $\mathbb{R}^d$. Then there exists a non-negative λ-integrable function p on $\mathbb{R}^d$ such that

$$f(\alpha) = \int_{\mathbb{R}^d} e^{i(t,\alpha)} p(t) d\lambda(t), \quad \alpha \in \mathbb{R}^d$$

if and only if f is non-negative definite continuous and

$$f(0) = \int_{\mathbb{R}^d} d\lambda(t) \lim_{\epsilon \to 0} \int_{\mathbb{R}^d} e^{-i(t,\alpha)} f(\alpha) C(\sqrt{\epsilon}\alpha) d\lambda(\alpha).$$

Proof. The necessity is evident from Lemma 1. We shall prove the sufficiency.

Assume that f satisfies the hypothesis. Then by the Bochner's theorem there exists a finite measure m on $\mathbb{R}^d$ such that

$$f(\alpha) = \int_{\mathbb{R}^d} e^{i(\alpha,t)} dm(t), \quad \alpha \in \mathbb{R}^d.$$

Let $dm(t) = p(t)d\lambda(t) + d\nu(t)$ be the Lebesgue decomposition of m where ν is singular to $d\lambda$. Then we have

$$f(\alpha) = \int e^{i(\alpha,t)} p(t)d\lambda(t) + \int e^{i(\alpha,t)} d\nu(t) = f_1(\alpha) + f_2(\alpha)$$

and by Lemma 1

$$p(t) = \lim_{\varepsilon \to 0} \int e^{i(t,\alpha)} f_1(\alpha) C(\sqrt{\varepsilon}\alpha) d\lambda(\alpha), \quad \text{a.s. } (d\lambda(t)).$$

On the other hand by Lemma 3, since ν is singular to λ, we have

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} e^{i(t,\alpha)} f_2(\alpha) C(\sqrt{\varepsilon}\alpha) d\lambda(\alpha)$$

$$= \lim_{\varepsilon} \int_{\mathbb{R}^d} \sqrt{\varepsilon}^{-d} C(\frac{s-t}{\sqrt{\varepsilon}}) d\nu(s) = 0, \quad \text{a.s. } d\lambda(t).$$

Therefore we have

$$p(t) = \lim_{\varepsilon} \int e^{i(t,\alpha)} f(\alpha) C(\sqrt{\varepsilon}\alpha) d\lambda(\alpha), \quad \text{a.s. } d\lambda(t).$$

Furthermore we have

$$\nu(\mathbb{R}^d) + \int_{\mathbb{R}^d} p(t)d\lambda(t) = m(\mathbb{R}^d) = f(0) = \int_{\mathbb{R}^d} p(t)d\lambda(t)$$

so that $\nu = 0$.   Q.E.D.

Now we can give a characterization of a μ-continuous characteristic functional in the case of tame functionals.

For a μ-measurable functional ψ on E and P in P define

$$(\overline{F}_P \psi)(x) = \lim_{\varepsilon \to 0} \overline{C}(Px) \mathbb{E}_\xi [e^{-i(Px,P\xi)} \psi(P\xi) C(\sqrt{\varepsilon}P\xi)]$$

if it has a meaning. In particular if $\psi(P\xi)$ is μ-integrable, then we have

$$(\overline{F}_P \psi)(x) = \overline{C}(Px) \mathbb{E}_\xi [e^{-i(Px,P\xi)} \psi(P\xi)].$$

THEOREM 3. Let ψ be a tame functional on H such that $\psi(\xi) = \psi(P\xi)$ for a P in P. Then $f(\xi) = C(\xi)\psi(\xi)$ is a μ-continuous characteristic functional if and only if $f(\xi)$ is non-negative definite continuous on H (so that $F_P \psi$ is well-defined) and we have $\mathbb{E}_x [(F_P \psi)(x)] = 1$.

And then $(\overline{F}_p\psi)(x)$ is the Radon-Nikodym derivative.

Proof. Assume that ψ satisfies the hypothesis. Then we have

$$(\overline{F}_p\psi)(x) = \lim_{\varepsilon \to 0} \overline{C}(Px)\, \mathbb{E}_\xi [e^{-i(Px,P\xi)}\psi(P\xi)C(\sqrt{\varepsilon}P\xi)]$$

$$= \lim_\varepsilon \overline{C}(Px)\, \int e^{-i(Px,P\xi)}\psi(P\xi)C(P\xi)C(\sqrt{\varepsilon}P\xi)d\lambda(P\xi)$$

$$= \lim_\varepsilon \overline{C}(Px)\, \int e^{-i(Px,P\xi)}f(P\xi)C(\sqrt{\varepsilon}P\xi)d\lambda(P\xi).$$

Since $f(P\xi)$ satisfies the hypothesis of Lemma 4 we have

$$f(P\xi) = \int e^{i(Px,P\xi)}(\overline{F}_p\psi)(x)C(Px)d\lambda(Px)$$

$$= \mathbb{E}_x[e^{i(Px,P\xi)}(\overline{F}_p\psi)(x)].$$

On the other hand by the definition we have $(\overline{F}_p\psi)(x) = (\overline{F}_p\psi)(Px)$, a.s. (μ), so that

$$f(\xi) = \overline{C}((I-P)\xi)f(P\xi)$$

$$= \overline{C}((I-P)\xi)\,\mathbb{E}_x[e^{i(Px,P\xi)}(\overline{F}_p\psi)(Px)]$$

$$= \mathbb{E}_x[e^{i(x,\xi)}(\overline{F}_p\psi)(x)].$$

The necessity is also proved in the same manner.

Q.E.D.

§4. GENERAL INVERSION FORMULA

THEOREM 4 (Inversion formula). Let ϕ be a μ-integrable function on E.

Then for an arbitrary approximation sequence $\{P_n\}$ in P we have

$$\overline{F}_{P_n} \hat{\phi} = \mathbb{E}_x[\phi | \mathcal{B}(P_n)] , \quad n \in \mathbb{N} .$$

Therefore $\overline{F}_{P_n} \hat{\phi}$ converges to ϕ simultaneously in $L^1(\mu)$ and almost surely.

Proof. By the definition we have

$$(\overline{F}_{P_n} \hat{\phi})(x)$$

$$= \lim_{\varepsilon \to 0} \overline{C}(P_n x) \mathbb{E}_\xi [e^{-i(P_n x, P_n \xi)} \hat{\phi}(P_n \xi) C(\sqrt{\varepsilon} P_n \xi)]$$

$$= \lim_{\varepsilon \to 0} \overline{C}(P_n x) \mathbb{E}_\xi [e^{-i(P_n x, P_n \xi)} \overline{C}(P_n \xi) \mathbb{E}_y [e^{i(y, P_n \xi)} \phi(y)] C(\sqrt{\varepsilon} P_n \xi)]$$

$$= \lim_{\varepsilon} \overline{C}(P_n x) \mathbb{E}_\xi [e^{-i(P_n x, P_n \xi)} \overline{C}(P_n \xi) \mathbb{E}_y [e^{i(P_n y, P_n \xi)} \mathbb{E}_y [\phi | \mathcal{B}(P_n)]] C(\sqrt{\varepsilon} P_n \xi)]$$

$$= \lim_{\varepsilon} \overline{C}(P_n x) \int e^{-i(P_n x, P_n \xi)} C(\sqrt{\varepsilon} P_n \xi) d\lambda(P_n \xi) \int e^{i(P_n y, P_n \xi)} (P_n \phi)(P_n y) C(P_n y) d\lambda(P_n y)$$

$$= (P_n \phi)(P_n x)$$

$$= \mathbb{E}_x[\phi | \mathcal{B}(P_n)], \quad \text{a.e.}$$

by Lemma 1.   Q.E.D.

THEOREM 5. Let ψ be a continuous functional on H such that $\psi(P_n \xi)$, $n \in \mathbb{N}$,

are μ-integrable for an approximation sequence $\{P_n\}$ in P. Then there exists

a μ-integrable function ϕ such that $\psi = F\phi$ if and only if $\overline{F}_{P_n} \psi$, $n \in \mathbb{N}$,

are μ-integrable and converge to ϕ in $L^1(\mu)$.

Proof. Assume ψ satisfies the hypothesis so that $\overline{F}_{P_n} \psi_n$ converges to a

μ-integrable function ϕ in $L^1(\mu)$. Then we have for every ξ in H

$$(F\phi)(\xi) = \overline{C}(\xi)\, \mathbb{E}_X[e^{i(x,\xi)}\phi(x)]$$

$$= \lim_n \overline{C}(\xi)\, \mathbb{E}_X[e^{i(x,\xi)}(\overline{F}_{P_n}\psi)(x)]$$

$$= \lim_n F\overline{F}_{P_n}\psi.$$

Since $(\overline{F}_{P_n}\psi)(x) = (\overline{F}_{P_n}\psi)(P_n x)$, a.s. (μ), by Theorem 2 we have

$$\psi(P_n\xi) = F\overline{F}_{P_n}\psi \to \psi(\xi) \text{ as } n \to \infty, \; \xi\epsilon H.$$

The necessity was proved in Theorem 4.

<div align="center">Q.E.D.</div>

Before stating Lemma 5, we remark that by the continuous injection of E into $(E^*)^a$, the algebraic dual of E^* with the weak topology $\sigma((E^*)^a, E^*)$, μ is extended to a Gaussian Radon measure $\overline{\mu}$ on $(E^*)^a$. On the other hand, every non-negative definite continuous functional f on H uniquely defines a probability measure $\overline{\nu}_f$ on $((E^*)^a, C((E^*)^a, E^*))$ such that f is the characteristic function of $\overline{\nu}_f$. Then it is easy to show that f is a μ-continuous characteristic functional on H if and only if $\overline{\nu}_f \ll \overline{\mu}$ on $((E^*)^a, C((E^*)^a, E^*))$.

LEMMA 5. Let f be a non-negative definite continuous functional on H, $\overline{\nu}_f$ be the corresponding probability measure on $((E^*)^a, C((E^*)^a, E^*))$ and $\{P_n\}$ be an arbitrary approximation sequence in P. Then if $\overline{\nu}_f$ is singular to $\overline{\mu}$ on $C((E^*)^a, E^*)$, then $(\overline{F}_{P_n}\psi)(x)$ converges to zero almost surely $(\overline{\mu})$, so that almost surely (μ), where $\psi(\xi) = \overline{C}(\xi)f(\xi)$, $\xi\epsilon H$.

Proof. Since E is a $\overline{\mu}$-measurable full set in $(E^*)^a$, random linear maps on E are naturally extended to $(E^*)^a$. Denote the σ-algebra on $(E^*)^a$ generated by $\{(x,\xi); \xi\epsilon P_n H\}$ by $\overline{B}_n$, $\underset{n}{\vee}\overline{B}_n$ by $\overline{B}_\infty$, the restriction of $\overline{\nu}_f$ to $\overline{B}_n$ by $\overline{\nu}_n$, and the restriction of $\overline{\mu}$ to $\overline{B}_n$ by $\overline{\mu}_n$, respectively. Then for every $C((E^*)^a, E^*)$-measurable function ϕ on $(E^*)^a$ there exists a $\overline{B}_\infty$-measurable function ϕ_1 such that $\phi = \phi_1$ a.s. $(\overline{\mu})$.

Let

$$\bar{\nu}_n(A) = \int_A p_n(x)d\bar{\mu}_n(x) + \bar{\theta}_n(A), \qquad A \in \bar{B}_n, \quad 1 \le n \le +\infty,$$

be the Lebesque decomposition of $\bar{\nu}_n$ with respect to $\bar{\mu}_n$ so that $\bar{\theta}_n$ is singular to $\bar{\mu}_n$. Since $\{\bar{B}_n\}$ is increasing, by J. Neveu [5] Proposition III-2-7 $\{p_n\}$ is a non-negative supermartingale so that P_n converges almost surely $(\bar{\mu})$ to p_∞. On the other hand $\bar{\nu}_f$ is singular to $\bar{\mu}$ so we have $p_\infty = 0$, a.s. $(\bar{\mu})$.

On the other hand we have

$$f(P_n\xi) = \int e^{i(x,P_n\xi)} p_n(x)d\bar{\mu}_n(x) + \int e^{i(x,P_n\xi)} d\bar{\theta}_n(x)$$

$$= f_1(P_n\xi) + f_2(P_n\xi), \qquad \xi \in H,$$

so that by the proof of Lemma 4 we have

$$\bar{F}_{P_n}\psi_1 = p_n(x), \quad \text{a.s.} (\bar{\mu}_n),$$

$$\bar{F}_{P_n}\psi_2 = 0, \quad \text{a.s.} (\bar{\mu}_n),$$

where we define $\psi_1 = \bar{C}f_1$ and $\psi_2 = \bar{C}f_2$. Therefore we have

$$\lim_n \bar{F}_{P_n}\psi = \lim_n \bar{F}_{P_n}\psi_1 + \lim_n \bar{F}_{P_n}\psi_2$$

$$= \lim_n p_n(x) = 0, \quad \text{a.s.} (\bar{\mu}).$$

$$\text{Q.E.D.}$$

THEOREM 6. Let f be a functional on H. Then f is a μ-continuous characteristic functional if and only if f is non-negative definite continuous on H and for a (so that any) approximation sequence P_n in P

$$\phi_n(x) = \bar{F}_{P_n}(\bar{C}f)(x)$$

converges in $L^1(\mu)$ to ϕ such that $E_x[\phi(x)] = 1$.

And then ϕ is the Radon-Nikodym derivative.

Proof. Assume that f is a μ-continuous characteristic functional with the Radon-Nikodym derivative p. Then we have $\overline{C}f = Fp$ so that by Theorem 4

$$\overline{F}_{P_n}(\overline{C}f) = \mathbb{E}_x[p|\mathcal{B}(P_n)]$$

which converges to p in $L^1(\mu)$ as n→∞.

Conversely assume that f satisfies the hypothesis of the theorem. Then there exists a probability measure $\overline{\nu}_f$ on $((E^*)^a, \mathcal{C}((E^*)^a, E^*))$ with the characteristic functional f. Let $d\overline{\nu}_f(x) = p(x)d\overline{\mu}(x) + d\overline{\theta}(x)$ be the Lebesque decomposition of $\overline{\nu}_f$ with respect to $\overline{\mu}$, the extension of μ to $(E^*)^a$. Then we have

$$f(\xi) = \int e^{i(x,\xi)}p(x)d\overline{\mu}(x) + \int e^{i(x,\xi)}d\overline{\theta}(x)$$

$$= f_1(\xi) + f_2(\xi), \qquad \xi \in H.$$

By Theorem 4 $\overline{F}_{P_n}(\overline{C}f_1)$ converges to p in $L^1(\overline{\mu})$. On the other hand, since $\overline{\theta}$ is singular to $\overline{\mu}$, by Lemma 5 either $\overline{F}_{P_n}(\overline{C}f_2)$ converges to zero in $L^1(\overline{\mu})$ or does not converge in $L^1(\overline{\mu})$. By hypothesis $\overline{F}_{P_n}(\overline{C}f) = \overline{F}_{P_n}(\overline{C}f_1) + \overline{F}_{P_n}(\overline{C}f_2)$ converges in $L^1(\overline{\mu})$ so that converges to p and we have $\int p(x)d\overline{\mu}(x) = 1 = \overline{\nu}_f((E^*)^a)$. Therefore $\overline{\theta} = 0$ and the theorem is proved.   Q.E.D.

Reformulating Theorem 6, we have the following proposition.

PROPOSITION 1. Let f be the characteristic functional of a Radon measure μ_1 on E. Then $\mu_1 \ll \mu$ if and only if f is continuous on H and for a (so that any) approximation sequence $\{P_n\}$ in P

$$\phi_n(x) = \lim_{\epsilon \to 0} \epsilon^{-d(n)} \int_E \exp\left[-\frac{1}{2\epsilon^2}|P_n x - P_n y|^2 + \frac{1}{2}|P_n x|^2\right]d\mu_1(y), \quad n \in \mathbb{N},$$

where d(n) is the dimension of $P_n H$, converges in $L^1(\mu)$ to ϕ such that $\mathbb{E}_x[\phi(x)] = 1$.

And then ϕ is the Radon-Nikodym derivative.

Proof. Since we have

$$f(\xi) = \int_E e^{i(y,\xi)} d\mu_1(y), \quad \xi \in H,$$

Fubini's theorem implies that

$$\overline{F}_{P_n}(\overline{C}f)(x)$$

$$= \lim_{\varepsilon \to 0} \overline{C}(P_n x) E_\xi [e^{-i(P_n x, P_n \xi) - \frac{\varepsilon^2}{2}|P_n \xi|^2} \overline{C}(P_n \xi) f(P_n \xi)]$$

$$= \lim_{\varepsilon \to 0} \overline{C}(P_n x) \int e^{-i(P_n x, P_n \xi) - \frac{\varepsilon^2}{2}|P_n \xi|^2} d\lambda(P_n \xi) \int_E e^{i(y, P_n \xi)} d\mu_1(y)$$

$$= \lim_{\varepsilon \to 0} \varepsilon^{-d(n)} \int_E \exp[-\frac{1}{2\varepsilon^2}|P_n x - P_n y|^2 + \frac{1}{2}|P_n x|^2] d\mu_1(y).$$

Then Theorem 6 proves the proposition.   Q.E.D.

Let $\Delta_n = \{e_{n,1}, e_{n,2}, \ldots, e_{n,d(n)}\}$, $n \in \mathbb{N}$, be a sequence of ONS in H. Then $\{\Delta_n\}$ is called an _approximation sequence of_ ONS _in_ E* iff we have $\Delta_n \subset E^*$, $\Delta_n \subset \Delta_{n+1}$, $n \in \mathbb{N}$ and the closed linear span of $\underset{n}{\cup}\Delta_n$ coincides with H. Let $\{\Delta_n\}$ be an approximation sequence of ONS in E*. Then for every n the projection

$$P_n x = \sum_{k=1}^{d(n)} \langle x, e_{n,k} \rangle e_{n,k}$$

is well-defined for all $x \in E$ and it is easy to show that $\{P_n\}$ is an approximation sequence in P. A slight modification of Proposition 1 gives the following proposition.

PROPOSITION 2. _Let_ μ_1 _be a Radon measure on_ E. _Then_ $\mu_1 \ll \mu$ _if and only if there exists an approximation sequence of_ ONS _in_ E*, _say_ $\Delta_n = \{e_{n,1}, e_{n,2}, \ldots, e_{n,d(n)}\}$, $n \in \mathbb{N}$, _such that_

$$\phi_n(x) = \lim_{\varepsilon \to 0} \varepsilon^{-d(n)} \int_E \exp[-\frac{1}{2\varepsilon^2} \sum_{k=1}^{d(n)} \{(1-\varepsilon^2)\langle x, e_{n,k} \rangle^2$$

$$- 2\langle x, e_{n,k} \rangle \langle y, e_{n,k} \rangle + \langle y, e_{n,k} \rangle^2] d\mu_1(y)$$

<u>converges in</u> $L^1(\mu)$ <u>to</u> ϕ <u>such that</u> $E_x[\phi(x)] = 1$.

And then ϕ <u>is the Radon-Nikodym derivative.</u>

Proof. Let P_n be the orthogonal projection of H onto the linear span of Δ_n, $n \in \mathbb{N}$. Then $\{P_n\}$ is an approximation sequence in P and we have

$$-\frac{1}{2\varepsilon^2}|P_n x - P_n y|^2 + \frac{1}{2}|P_n x|^2$$

$$= \frac{1}{2\varepsilon^2} \sum_{k=1}^{d(n)} \{(1-\varepsilon^2)<x,e_{n,k}>^2 - 2<x,e_{n,k}><y,e_{n,k}> + <y,e_{n,k}>^2\} \ ,$$

which is well-defined for all x,y in E since $\Delta_n \subset E^*$, $n \in \mathbb{N}$. Therefore, without using the continuity of $\tilde{\mu}_1(\xi)$ on H, from the hypothesis we can construct the Radon-Nikodym derivative and the sufficiency is proved.

The necessity is proved as a simple corollary of Proposition 1.

Q.E.D.

We apply Proposition 2 to the Wiener measure.

Let B(t), $0 \le t \le 1$, be the standard Brownian motion, X(t) be an arbitrary continuous process on [0,1] vanishing at zero. Then B(t) and X(t) induce the Wiener measure μ and a Radon measure μ_1 on the real separable Banach space C[0,1], the space of all continuous functions vanishing at zero, respectively. The topological dual space of C[0,1] is $\mu(0,1]$, the space of all signed measures on (0,1].

Let

$$\mathcal{D}_n: \ 0 = t_{n,0} < t_{n,1} < \ldots < t_{n,d(n)} = 1$$

be a decomposition of [0,1] such that $\mathcal{D}_{n+1}$ is a refinement of $\mathcal{D}_n$, $n \in \mathbb{N}$. Let $\delta(t)$ be the Dirac measure concentrated on t [0,1],

$$e_{n,k} = |\Delta_{n,k}|^{-\frac{1}{2}}\{\delta(t_{n,k}) - \delta(t_{n,k-1})\}, \ 1 \le k \le d(n), \ n \in \mathbb{N} \ ,$$

where $|\Delta_{n,k}| = t_{n,k} - t_{n,k-1}$, and assume that $\lim_{n} \max_{k} |\Delta_{n,k}| = 0$. Then it is

not difficult to show that $\Delta_n = \{e_{n,1}, e_{n,2}, \ldots, e_{n,d(n)}\}$, $n \in N$, is an approximation sequence of ONS in $\mu(0,1]$. As a corollary of Proposition 2 we have the following theorem.

THEOREM 7. <u>Let</u> $B(t)$, $0 \leq t \leq 1$, <u>be the standard Brownian motion</u>, $X(t)$, $0 \leq t \leq 1$, <u>be an arbitrary continuous process vanishing at</u> zero, μ <u>be the Wiener measure induced by</u> $B(t)$ <u>and</u> μ_1 <u>be the Radon measure induced by</u> $X(t)$ <u>on</u> $C[0,1]$. <u>Then</u> $\mu_1 \ll \mu$ <u>if and only if</u>

$$\phi_n = \lim_{\varepsilon \to 0} \varepsilon^{-d(n)} \mathbf{E}^x \left| \exp -\frac{1}{2\varepsilon^2} \sum_{k=1}^{d(n)} \frac{1}{|\Delta_{n,k}|} \{(1-\varepsilon^2) B(\Delta_{n,k})^2 - 2B(\Delta_{n,k}) X(\Delta_{n,k}) + X(\Delta_{n,k})^2\}) \right|,$$

<u>converges in</u> $L^1(\mu)$ <u>to</u> ϕ <u>such that</u> $\mathbf{E}^B[\phi] = 1$, <u>where</u> $B(\Delta_{n,k}) = B(t_{n,k}) - B(t_{n,k-1})$, $X(\Delta_{n,k}) = X(t_{n,k}) - X(t_{n,k-1})$, $1 \leq k \leq d(n)$, $n \in \mathbb{N}$, <u>and</u> $\mathbf{E}^x[\ ]$ <u>and</u> $\mathbf{E}^B[\ ]$ <u>are the expectations with respect to</u> $X(t)$ <u>and</u> $B(t)$, <u>respectively.</u>

<u>And then</u> ϕ <u>is the Radon-Nikodym derivative.</u>

§5. CONVOLUTION.

Let ν be a Radon measure on E and $\mu*\nu$ be the convolution of μ and ν. Then the characteristic functional is given by $\widetilde{\mu*\nu}(\xi) = \tilde{\mu}(\xi)\tilde{\nu}(\xi) = C(\xi)\tilde{\nu}(\xi)$, $\xi\epsilon E^*$, where $\tilde{\nu}$ is the characteristic functional of ν. If $\mu*\nu \ll \mu$, then $\tilde{\nu}(\xi)$ is continuous on H. The aim of this paragraph is to give some sufficient conditions on ν for $\mu*\nu \ll \mu$.

LEMMA 6.

(1) If $\nu=\delta_x$, the Dirac measure concentrated on $x\epsilon E$, then $\mu*\delta_x \ll \mu$ if and only if $x\epsilon H$.

(2) If $\nu(H)=1$, then $\mu*\nu \ll \mu$.

(3) Assume that $\mu*\nu \ll \mu$ and let E_0 be a full linear subspace of E. Then we have $\nu(E_0) = 1$.

Proof. (1) is well-known.

(2) Assume that $\nu(H) = 1$ and $\mu(N) = 0$. Then by (1) $\mu(N-x) = (\mu*\delta_x)(N) = 0$ for all $x\epsilon H$ so that we have

$$(\mu*\nu)(N) = \int_E\mu(N-x)d\nu(x)$$

$$= \int_H\mu(N-x)d\nu(x) = 0.$$

(3) Assume that $\mu*\nu \ll \mu$ and let E_0 be a full linear subspace. Then we have

$$1 = (\mu*\nu)(E_0) = \int_E\mu(E_0-x)d\nu(x),$$

so that $\nu(X\epsilon E: \mu(E_0-x) = 1) = 1$. Since E_0 is a linear subspace, we have $(E_0-x) \cap E_0 = \emptyset$ if $x\notin E_0$. Therefore we have $\{x\epsilon E; \mu(E_0-x) = 1\} \subset E_0$ and $\nu(E_0) \geq \nu(x\epsilon E: \mu(E_0-x) = 1) = 1$.   Q.E.D.

By Lemma 6 our interest is reduced to the case where $\nu(H) = 0$.

Since $\tilde{\nu}$ is bounded, Theorem 5 is applicable and we have the following proposition as a corollary.

PROPOSITION 3. <u>Let</u> ν <u>be a Radon measure on</u> E. <u>Then we have</u> $\mu*\nu \ll \mu$

<u>if and only if</u> $\tilde{\nu}(\xi)$ <u>is continuous on</u> H <u>and there exists an approximation</u>

<u>sequence</u> $\{P_n\}$ <u>in</u> P <u>such that</u>

$$\phi_n(x) = \int_E e^{(P_n x, P_n y) - \frac{1}{2}|P_n y|^2} d\nu(y),$$

<u>converges in</u> $L^1(\mu)$.

  <u>And then the limit function is the Radon-Nikodym derivative.</u>

Proof. Let P_n be an approximation sequence in P. Then, since $\{P_n\}$ is

bounded so that integrable for every $n \in \mathbb{N}$, we have

$$\phi_n(x) = \overline{F}_{P_n} \tilde{\nu}(x)$$

$$= C(P_n x) \mathbb{E}_\xi \left[e^{-i(P_n x, P_n \xi)} \int_E e^{i(P_n y, P_n \xi)} d\nu(y) \right]$$

$$= \int_E e^{(P_n x, P_n y) - \frac{1}{2}|P_n y|^2} d\nu(y),$$

which is μ-integrable for $C(\xi)\tilde{\nu}(\xi)$ is non-negative definite. Therefore Theorem

5 proves the theorem.

$$\text{Q.E.D.}$$

  As in Proposition 2 we can modify Theorem 7 as follows. The proof is

similar to that of Proposition 2.

PROPOSITION 4. <u>Let</u> ν <u>be a Radon measure on</u> E. <u>Then we have</u> $\mu*\nu \ll \mu$ <u>if and</u>

<u>only if there exists an approximation sequence of</u> ONS <u>in</u> E*, <u>say</u>

$\Delta_n = \{e_{n,1}, e_{n,2}, \ldots, e_{n,d(n)}\}$, $n \in \mathbb{N}$, <u>such that</u>

$$\phi_n(x) = \int_E \exp \left[\sum_{k=1}^{d(n)} <x, e_{n,k}><y, e_{n,k}> - \frac{1}{2}<y, e_{n,k}>^2 \right] d\nu(y)$$

<u>converges in</u> $L^1(\mu)$.

  <u>And then the limit function is the Radon-Nikodym derivative.</u>

Proposition 4 gives the following theorem in a similar manner to the proof of Theorem 7. We use the same notations with Theorem 7.

THEOREM 8. Let $B(t)$, $0 \leq t \leq 1$, be the standard Brownian motion, $X(t)$, $0 \leq t \leq 1$, be an arbitrary continuous process vanishing at 0 and independent of $\{B(t)\}$, μ be the Wiener measure induced by $B(t)$ and ν be the Radon measure induced by $X(t)$ on $C[0,1]$. Then $B(t) + X(t)$, $0 \leq t \leq 1$, induces $\mu * \nu$ and we have $\mu * \nu \ll \mu$ if and only if there exists a sequence of decomposition of $[0,1]$

$$\mathcal{D}_n: \quad 0 = t_{n,0} < t_{n,1} < \ldots < t_{n,k} < \ldots < t_{n,d(n)} = 1$$

such that $\mathcal{D}_{n+1}$ is a refinement of $\mathcal{D}_n$, $n \in \mathbb{N}$, $\lim_n \max_k |\Delta_{n,k}| = 0$ and

$$\phi_n = \mathbb{E}^X \left[\exp \left(\sum_{k=1}^{d(n)} \frac{1}{|\Delta_{n,k}|} \{ B(\Delta_{n,k}) X(\Delta_{n,k}) - \frac{1}{2} X(\Delta_{n,k})^2 \} \right) \right]$$

converges in $L^1(\mu)$.

The following theorems are also applications of Proposition 4.

THEOREM 9. Let ν be a Radon measure on E and assume that there exists a CONS $\{e_k\}$ of H chosen from E^* such that

$$\sup_n \int_{E \times E} \exp \left(\sum_{k=1}^n <y,e_k><y',e_k> \right) d\nu(y) d\nu(y') < +\infty.$$

Then $\mu * \nu \ll \mu$.

Proof. Obviously $\Delta_n = \{e_1, e_2, \ldots, e_n\}$, $n \in \mathbb{N}$, is an approximation sequence of ONS in E^* so that Proposition 4 is applicable and we have

$$\phi_n(x) = \int_E \exp \left(\sum_{k=1}^n <x,e_k><y,e_k> - \frac{1}{2}<y,e_k>^2 \right) d\nu(y).$$

On the other hand, since $\{<x,e_k>\}$ is a standard Gaussian sequence so that ϕ_n is a $B(P_n)$-martingale where P_n is the orthogonal projection of H to the linear span of $\{e_1, e_2, \ldots, e_n\}$.

Then

$$\sup_{n} E_X[\phi_n(x)^2]$$

$$= \sup_{n} \int_{E \times E} \exp \left(\sum_{k=1}^{n} <y,e_k><y',e_k> \right) d\nu(y) d\nu(y') < +\infty$$

implies the uniform integrability of $\{\phi_n\}$ and ϕ_n converges in $L^1(\mu)$.

<div align="right">Q.E.D.</div>

Before stating Theorem 10 we prove the following lemmas.

LEMMA 7. <u>Let</u> $\{X_k\}$ <u>be an independent real random sequence on</u> (E,μ) <u>such that the expectation</u> $E_X[X_k(x)] = 0$, $1 + X_k > 0$, a.s., $k \in \mathbb{N}$, <u>and</u> $Z = \lim_{n} \prod_{k=1}^{n} (1+X_k) > 0$, a.s.. <u>Then</u> $Z_n = \prod_{k=1}^{n} (1+X_k)$ <u>converges to</u> Z <u>in</u> $L^1(\mu)$.

Proof. Define $Y_k = (1+X_k)^{\frac{1}{2}}$, $k \in \mathbb{N}$. Then we have $E_X[Y_k] \leq 1$ and

$$\prod_{k=1}^{\infty} E_X[Y_k] = \lim_{n} \prod_{k=1}^{n} E_X[Y_k]$$

$$= \lim_{n} E_X \left[\prod_{k=1}^{n} Y_k \right] = \lim_{n} E_X[Z_n^{\frac{1}{2}}]$$

$$\geq E_X[\liminf_{n} Z_n^{\frac{1}{2}}] = E_X[Z^{\frac{1}{2}}] > 0.$$

Therefore we have for every $n,m \in \mathbb{N}$ ($n<m$)

$$E_X \left[|Z_n - Z_m| \right]^2$$

$$= E_X \left[|Z_n^{\frac{1}{2}} - Z_m^{\frac{1}{2}}| |Z_n^{\frac{1}{2}} + Z_m^{\frac{1}{2}}| \right]^2$$

$$\leq 4 E_X \left[|Z_n^{\frac{1}{2}} - Z_m^{\frac{1}{2}}|^2 \right]$$

$$= 8 E_X \left[1 - Z_n^{\frac{1}{2}} Z_m^{\frac{1}{2}} \right]$$

$$= 8 \{ 1 - E_X [\prod_{n<k\leq m} Y_k] \}$$

$$= 8 \{ 1 - \prod_{n<k\leq m} E_X[Y_k] \} \to 0 \quad \text{as } n,m \to \infty.$$

<div align="right">Q.E.D.</div>

LEMMA 8. <u>Let</u> $\{x_k\}$ <u>be a real sequence such that</u> $x_k > -1$, $k \epsilon N$. <u>If</u> Σx_k <u>and</u> Σx_k^2 <u>converge, then</u> $Z_n = \prod\limits_{k=1}^{n} (1+x_k)$ <u>converges to a positive number.</u>

Proof. Since Σx_k converges, we have $\lim\limits_{k} x_k = 0$ so that we may assume $|x_k| < 1$,

$k \epsilon N$. By the Taylor expansion we have

$$\sum_{k} \log (1+x_k) = \sum_{k} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x_k^n .$$

For every N,M $(N<M)$ ϵ N we have

$$\left| \sum_{N<k<M} \sum_{n\geq 2} \frac{(-1)^{n+1}}{n} x_k^n \right| \leq \sum_{n\geq 2} \frac{1}{n} \sum_{N<k<M} |x_k|^n$$

$$\leq \sum_{n\geq 2} \frac{1}{n} \left(\sum_{N<k<M} x_k^2 \right)^{\frac{n}{2}} \rightarrow 0 \text{ as } N,M \rightarrow +\infty$$

since $\Sigma x_n^2 < +\infty$. Therefore

$$\sum_{k} \log (1+x_k) = \sum_{k} x_k + \sum_{n\geq 2} \frac{(-1)^{n+1}}{n} \sum_{k} x_k^n$$

converges.   Q.E.D.

THEOREM 10. <u>Let</u> ν <u>be a Radon measure on</u> E <u>and assume that there exists a CONS</u> $\{e_k\}$ <u>of</u> H <u>chosen from</u> E* <u>such that</u> $\{<y,e_k>\}$ <u>is an independent symmetric random sequence on</u> (E,ν), <u>and</u>

$$M = \sum_{n=2}^{\infty} \frac{1}{2^n n!} \sum_{k} \int_E <y,e_k>^{2n} e^{-\frac{1}{2}<y,e_k>^2} d\nu(y) < +\infty.$$

<u>Then we have</u> $\mu * \nu \sim \mu$.

Proof. Assume that the hypothesis is satisfied. Since $\{<y,e_k>\}$ are independent, we have

$$\phi_n(x) = \int_E \prod_{k=1}^{n} \exp [<x,e_k><y,e_k> - \frac{1}{2}<y,e_k>^2] d\nu(y)$$

$$= \prod_{k=1}^{n} \int_E \exp \{<x,e_k><y,e_k> - \frac{1}{2}<y,e_k>^2\} d\nu(y), \qquad n \epsilon N.$$

Define

$$X_k = \int_E \exp\left[<x,e_k><y,e_k> - \frac{1}{2}<y,e_k>^2\right]d\nu(y) - 1, \quad k \in \mathbb{N}$$

Then $\{X_k\}$ is an independent random sequence on (E,μ) such that $\mathbb{E}_x[X_k(x)] = 0$

and $1+X_k > 0$, a.s. (μ), $k \in \mathbb{N}$. Therefore if we prove that

$$\prod_{k=1}^{\infty} (1+X_k) > 0, \text{ a.s. } (\mu),$$ then by Lemma 7 $\phi_n = \prod_{k=1}^{n} (1+X_k)$ converges in $L^1(\mu)$. By

Lemma 8 it is enough to prove the convergences of $\sum_k X_k$ and $\sum_k X_k^2$.

In fact we have by the symmetricity of $\{<y,e_k>\}$

$$X_k = \int_E \sum_{n=0}^{\infty} \frac{1}{n!} <x,e_k>^n <y,e_k>^n e^{-\frac{1}{2}<y,e_k>^2} d\nu(y) - 1$$

$$= \sum_{n=0}^{\infty} \frac{1}{(2n)!} <x,e_k>^{2n} \int_E <y,e_k>^{2n} e^{-\frac{1}{2}<y,e_k>} d\nu(y)$$

$$- \sum_{n=0}^{\infty} \int_E \frac{1}{n!} \frac{1}{2^{2n}} <y,e_k>^{2n} e^{-\frac{1}{2}<y,e_k>^2} d\nu(y)$$

$$= \frac{1}{2}\{<x,e_k>^2 - 1\}\int_E <y,e_k>^2 e^{-\frac{1}{2}<y,e_k>^2} d\nu(y)$$

$$+ \sum_{n=2}^{\infty} \{\frac{<x,e_k>^{2n}}{(2n)!} - \frac{1}{2^n n!}\} \int_E <y,e_k>^{2n} e^{-\frac{1}{2}<y,e_k>^2} d\nu(y)$$

$$= V_k + W_k.$$

Since $\{<x,e_k>\}$ is a standard Gaussian sequence on (E,μ) and by Schwarz'

inequality we have

$$\sum_k \left| \int_E <y,e_k>^2 e^{-\frac{1}{2}<y,e_k>^2} d\nu(y) \right|^2$$

$$\leq \sum_k \int_E <y,e_k>^4 e^{-\frac{1}{2}<y,e_k>^2} d\nu(y) \leq 2M < +\infty,$$

$\sum V_k$ and $\sum V_k^2$ converge almost surely.

On the other hand we have

$$\sum_k \mathbf{E}_x[|W_k|] \le \sum_k \sum_{n\ge 2} \left(\frac{1}{(2n)!} \, \mathbb{E}\,[<x,e_k>^{2n}] + \frac{1}{2^n n!}\right) \int_E <y,e_k>^{2n} \, e^{-\frac{1}{2}<y,e_k>^2} \, d\nu(y)$$

$$= 2 \sum_{n\ge 2} \frac{1}{2^n n!} \int_E <y,e_k>^{2n} \, e^{-\frac{1}{2}<y,e_k>^2} \, d\nu(y) \le 2M < +\infty.$$

Therefore ΣW_k converges absolutely a.s. (μ) so that ΣW_k^2 converges almost surely.

These imply that $\displaystyle\sum_k X_k = \sum_k V_k + \sum_k W_k$ and $\displaystyle\sum_k X_k^2 = \sum_k V_k^2 + 2\sum_k V_k W_k + \sum_k W_k^2$ converge

almost surely (μ).
<div style="text-align:right">Q.E.D.</div>

EXAMPLE 2. Let $E = \mathbf{R}^{\mathbf{N}}$, the space of all real sequences, μ is the product

measure of copies of $N(0,1)$. Then we have $H = \ell_2$. Let ν be the product

measure of $N(0,\sqrt{v_k})$, $v_k > 0$, $k\in\mathbb{N}$. Then $\mu*\nu \sim \mu$ if $\displaystyle\sum_k v_k^2 < +\infty$.

Remark that $\nu(\ell_2) = 0$ if $\Sigma v_k = +\infty$.

In fact let $\{e_k\}$ be the canonical CONS of ℓ_2. Then we have for every

$k,n \in \mathbb{N}$

$$\sum_{n=2}^{\infty} \frac{1}{2^n n!} \sum_k \int_E <y,e_k>^{2n} \, e^{-\frac{1}{2}<y,e_k>^2} \, d\nu(y)$$

$$= \sum_{k=1}^{\infty} \int \{e^{\frac{1}{2}<y,e_k>^2} - 1 - \frac{1}{2}<y,e_k>^2\} e^{-\frac{1}{2}<y,e_k>^2} d\nu(y) = \sum_{k=1}^{\infty} \{1 - \frac{1}{\sqrt{1+v_k}} - \frac{1}{2}\frac{v_k}{\sqrt{1+v_k}^3}\}$$

$$= \sum_{k=1}^{\infty} \{\frac{v_k^2}{2\sqrt{1+v_k}^3} \frac{2\sqrt{1+v_k} + 1}{(\sqrt{1+v_k} + 1)^2} \le \sum_k v_k^2 < +\infty.$$

COROLLARY. <u>Let ν be a Radon measure on E and assume that there exists a CONS</u>

<u>$\{e_k\}$ of H chosen from E* such that $\{<y,e_k>\}$ is an independent symmetric</u>

<u>random sequence on (E,ν) and</u>

$$\int \exp\left[\frac{1}{2} \left(\sum_k <y,e_k>^4\right)^{\frac{1}{2}}\right] d\nu(y) < +\infty.$$

<u>Then</u> $\mu*\nu \sim \mu$.

Proof. We have

$$
\sum_{n\geq 2} \frac{1}{2^n n!} \sum_{k=1}^{\infty} \int <y,e_k>^{2n} e^{-\frac{1}{2}<y,e_k>^2} d\nu(y) \leq \sum_{n\geq 2} \frac{1}{n!} \int \sum_k \left[\frac{<y,e_k>^2}{2}\right]^n d\nu(y)
$$

$$
\leq \sum_{n\geq 2} \frac{1}{n!} \int \left\{ \sum_k \frac{<y,e_k>^4}{4} \right\}^{\frac{n}{2}} d\nu(y) \leq \int \exp[\frac{1}{2}(\sum_k <y,e_k>^4)^{\frac{1}{2}}] d\nu(y) < +\infty.
$$

EXAMPLE 3.   Let μ be the same as in Example 2 and $\nu = \prod_n \frac{1}{2} \left(\delta_{a_n} + \delta_{-a_n}\right)$ where

$\sum_n a_n^4 < +\infty$.   Then $\mu * \nu \sim \mu$.

   $\nu(\ell_2) = 1$ if and only if $\sum_n a_n^2 < +\infty$.

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